# Derivatives Of The Cumulative Normal Distribution Function

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There are times in mathematical finance when we need the derivatives of the cumulative normal distribution function. In this white paper we will develop the mathematics to calculate the first and second derivatives of this function and solve the following hypothetical problem...

## Our Hypothetical Problem

We will use the hypothetical problem from the white paper [The Calculus of the Normal Distribution, Schurman, October 2010]. In this white paper we are pulling a random number from a normal distribution with a mean of 2.5 and a variance of 4.0. In this white paper we want to answer the following questions...

**Question 1**: What is the probability that our random number will be between 1.0 and 3.0?

**Question 2**: Using the answer to Question 1 above and the derivatives of the cumulative normal distribution function what is the probability that our random number will be between 1.0 and 3.1?

**Question 3**: Using the answer to Question 1 above and the derivatives of the cumulative normal distribution function what is the probability that our random number will be between 0.9 and 3.0?

## The Cumulative Normal Distribution Function

The probability density function (PDF) measures the height of the curve at any point $x$ that lies within the probability distribution's domain. We will define the function $f(x,m,v)$ to be the probability density function of the normal distribution with mean $m$ and variance $v$ at some point $x$. The equation for the normal distribution's probability density function is...

$$f(x,m,v)=\sqrt{\frac{1}{2\pi v}}\,\mathrm{Exp}\left\{-\frac{1}{2v}\left(x-m\right)^2\right\} \tag{1}$$

We will define the function $g(z,m,v,a,b)$ to be the cumulative normal distribution function, which is the probability that random variable $z$ pulled from a normal distribution with mean $m$ and variance $v$ has a value that lies within the range $[a,b]$. Using Equation (1) above the equation for the cumulative normal distribution function is...

$$g(z,m,v,a,b)=\mathrm{Prob}\left[a<z<b\right]=\int\limits_a^b f(x,m,v)\,\delta x=\int\limits_a^b\sqrt{\frac{1}{2\pi v}}\,\mathrm{Exp}\left\{-\frac{1}{2v}\left(x-m\right)^2\right\}\delta x \tag{2}$$

Note that we can write Equation (2) above as the difference between two cumulative normal distribution functions where the equation integral's lower bound is negative infinity...

$$g(z,m,v,a,b)=\int\limits_{-\infty}^b f(x,m,v)\,\delta x-\int\limits_{-\infty}^a f(x,m,v)\,\delta x=g(z,m,v,-\infty,b)-g(z,m,v,-\infty,a) \tag{3}$$

Note that using standard Excel functions we can rewrite Equation (3) above as...

$$g(z,m,v,a,b)=\mathrm{NORM.DIST(b,m,}\sqrt{v}\mathrm{,TRUE)}-\mathrm{NORM.DIST(a,m,}\sqrt{v}\mathrm{,TRUE)} \tag{4}$$

## Upper and Lower Bound Derivatives

Using Appendix Equation (16) below the first derivative of the cumulative normal distribution function Equation (2) above with respect to the upper bound of integration ($b$) is...

$$\frac{\delta}{\delta b}\,g(z,m,v,a,b)=\frac{\delta}{\delta b}\left(\int\limits_a^b\sqrt{\frac{1}{2\pi v}}\,\mathrm{Exp}\left\{-\frac{1}{2v}\left(x-m\right)^2\right\}\delta x\right)=\sqrt{\frac{1}{2\pi v}}\,\mathrm{Exp}\left\{-\frac{1}{2v}\left(b-m\right)^2\right\} \tag{5}$$

Using Appendix Equation (20) below the equation for the second derivative of the cumulative normal distribution function Equation (2) above with respect to the upper bound of integration ($b$) is...

$$\frac{\delta^2}{\delta b^2}\,g(z,m,v,a,b)=\frac{\delta}{\delta b}\left(\sqrt{\frac{1}{2\pi v}}\,\mathrm{Exp}\left\{-\frac{1}{2v}\left(b-m\right)^2\right\}\right)=\frac{m-b}{v}\sqrt{\frac{1}{2\pi v}}\,\mathrm{Exp}\left\{-\frac{1}{2v}\left(b-m\right)^2\right\} \tag{6}$$

Using Appendix Equation (27) below the first derivative of the cumulative normal distribution function Equation (2) above with respect to the lower bound of integration ($a$) is...

$$\frac{\delta}{\delta a}\,g(z,m,v,a,b)=\frac{\delta}{\delta a}\left(\int\limits_a^b\sqrt{\frac{1}{2\pi v}}\,\mathrm{Exp}\left\{-\frac{1}{2v}\left(x-m\right)^2\right\}\delta x\right)=-\sqrt{\frac{1}{2\pi v}}\,\mathrm{Exp}\left\{-\frac{1}{2v}\left(a-m\right)^2\right\} \tag{7}$$

Using Appendix Equation (29) below the equation for the second derivative of the cumulative normal distribution function Equation (**??**) above with respect to the lower bound of integration ($a$) is......

$$\frac{\delta^2}{\delta a^2}\,g(z,m,v,a,b)=\frac{\delta}{\delta a}\left(-\sqrt{\frac{1}{2\pi v}}\,\mathrm{Exp}\left\{-\frac{1}{2v}\left(a-m\right)^2\right\}\right)=\frac{a-m}{v}\sqrt{\frac{1}{2\pi v}}\,\mathrm{Exp}\left\{-\frac{1}{2v}\left(a-m\right)^2\right\} \tag{8}$$

## The Answers to Our Hypothetical Problem

**Question 1**: What is the probability that our random number will be between 1.0 and 3.0?

Using Equation (4) above the answer to this problem is...

$$\begin{aligned}\mathrm{Prob}\left[1.0<z<3.0\right]&=\text{NORM.DIST(3,2.5,}\sqrt{4}\text{,TRUE)}-\text{NORM.DIST(1,2.5,}\sqrt{4}\text{,TRUE)}\\&=0.5987-0.2266\\&=0.3721\end{aligned} \tag{9}$$

**Question 2**: Using the answer to Question 1 above and the derivatives of the cumulative normal distribution function what is the probability that our random number will be between 1.0 and 3.1?

Step 1: Using Equation (5) above...

$$\frac{\delta}{\delta b}\,g(z,2.5,4,1,3)=\sqrt{\frac{1}{2\times\pi\times4.0}}\,\mathrm{Exp}\left\{-\frac{1}{2\times4.0}\left(3.0-2.5\right)^2\right\}=0.1933 \tag{10}$$

Step 2: Using Equation (6) above...

$$\frac{\delta^2}{\delta b^2}\,g(z,2.5,4,1,3)=\frac{2.5-3.0}{4.0}\sqrt{\frac{1}{2\times\pi\times4.0}}\,\mathrm{Exp}\left\{-\frac{1}{2\times4.0}\left(3.0-2.5\right)^2\right\}=-0.0242 \tag{11}$$

Using Equations (10) and (11) above the answer to Question 2 is...

$$\begin{aligned}\mathrm{Prob}\left[1.0<z<3.1\right]&=\mathrm{Prob}\left[1.0<z<3.0\right]+\frac{\delta}{\delta b}\,g(z,m,v,a,b)\,\delta b+\frac{1}{2}\,\frac{\delta^2}{\delta b^2}\,g(z,m,v,a,b)\,\delta b^2\\&=0.3721+0.1933\times0.10+0.50\times-0.0242\times0.10^2\\&=0.3913\end{aligned} \tag{12}$$

**Question 3**: Using the answer to Question 1 above and the derivatives of the cumulative normal distribution function what is the probability that our random number will be between 0.9 and 3.0?

Step 1: Using Equation (7) above...

$$\frac{\delta}{\delta a}\,g(z,2.5,4,1,3)=-\sqrt{\frac{1}{2\times\pi\times 4.0}}\,\text{Exp}\left\{-\frac{1}{2\times 4.0}\left(1.0-2.5\right)^{2}\right\}=-0.1506 \tag{13}$$

Step 2: Using Equation (8) above...

$$\frac{\delta^2}{\delta a^2}\,g(z,2.5,4,1,3)=\frac{1.0-2.5}{4.0}\sqrt{\frac{1}{2\times\pi\times 4.0}}\,\text{Exp}\left\{-\frac{1}{2\times 4.0}\left(1.0-2.5\right)^{2}\right\}=-0.0565 \tag{14}$$

Using Equations (13) and (14) above the answer to Question 3 is...

$$\begin{aligned}\text{Prob}\left[0.9<z<3.0\right]&=\text{Prob}\left[1.0<z<3.0\right]+\frac{\delta}{\delta a}\,g(z,m,v,a,b)\,\delta a+\frac{1}{2}\,\frac{\delta^2}{\delta a^2}\,g(z,m,v,a,b)\,\delta a^2\\&=0.3721-0.1506\times -0.10+0.50\times -0.0565\times -0.10^2\\&=0.3869\end{aligned} \tag{15}$$

## Appendix

**A.** The equation for the first derivative of Equation (2) above with respect to the upper bound of integration $b$ is...

$$\frac{\delta}{\delta b}\,g(z,m,v,a,b)=\frac{\delta}{\delta b}\left(\int\limits_a^b\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\left(x-m\right)^2\right\}\delta x\right)=\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\left(b-m\right)^2\right\} \tag{16}$$

**B.** Using Equation (16) above the equation for the second derivative of Equation (2) above with respect to the upper bound of integration $b$ is...

$$\frac{\delta^2}{\delta b^2}\,g(z,m,v,a,b)=\frac{\delta}{\delta b}\left(\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\left(b-m\right)^2\right\}\right) \tag{17}$$

To solve Equation (17) above we will make the following simplifying definitions...

$$\Gamma_1=\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{\Gamma_2\right\}\ \text{...and...}\ \Gamma_2=-\frac{1}{2v}\,\Gamma_3^2\ \text{...and...}\ \Gamma_3=b-m \tag{18}$$

The derivatives of the definitions in Equation (18) above are...

$$\frac{\delta\Gamma_1}{\delta\Gamma_2}=\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{\Gamma_2\right\}\ \text{...and...}\ \frac{\delta\Gamma_2}{\delta\Gamma_3}=-\frac{1}{v}\,\Gamma_3\ \text{...and...}\ \frac{\delta\Gamma_3}{\delta b}=1 \tag{19}$$

Using Equations (18) and (19) above we can rewrite Equation (17) above as...

$$\frac{\delta^2}{\delta b^2}\,g(z,m,v,a,b)=\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{\Gamma_2\right\}\times -\frac{1}{v}\,\Gamma_3\times 1=\frac{m-b}{v}\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\left(b-m\right)^2\right\} \tag{20}$$

**C.** The equation for the first derivative of Equation (2) above with respect to the lower bound of integration $a$ is...

$$\frac{\delta}{\delta a}\,g(z,m,v,a,b)=\frac{\delta}{\delta a}\left(\int\limits_a^b\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\left(x-m\right)^2\right\}\delta x\right) \tag{21}$$

To solve Equation (21) above we will make the following simplifying definitions...

$$\theta=-x\ \text{..where...}\ \frac{\delta\theta}{\delta x}=-1\ \text{...such that...}\ -\delta\theta=\delta x \tag{22}$$

Using the definitions in Equation (22) above we can rewrite Equation (21) above as...

$$\frac{\delta}{\delta a}\,g(z,m,v,a,b)=\frac{\delta}{\delta a}\Bigg(\int\limits_{-a}^{-b}\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\bigg(-\theta-m\bigg)^2\right\}(-\delta\theta)\Bigg) \tag{23}$$

We can switch the bounds of integration and rewrite Equation (23) above as...

$$\frac{\delta}{\delta a}\,g(z,m,v,a,b)=\frac{\delta}{\delta a}\Bigg(\int\limits_{-b}^{-a}\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\bigg(-\theta-m\bigg)^2\right\}\delta\theta\Bigg) \tag{24}$$

To solve Equation (24) above we will define the temporary variable $c$ to be...

$$c=-a\ \text{ ...such that... }\ \frac{\delta c}{\delta a}=-1 \tag{25}$$

Using Equation (25) above we can rewrite Equation (24) above as...

$$\frac{\delta}{\delta a}\,g(z,m,v,a,b)=\frac{\delta}{\delta c}\Bigg(\int\limits_{-b}^{c}\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\bigg(-\theta-m\bigg)^2\right\}\delta\theta\Bigg)\times\frac{\delta c}{\delta a}=-\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\bigg(-c-m\bigg)^2\right\} \tag{26}$$

After substituting for the temporary variable $c$ in Equation (26) above the equation for the first derivative of Equation (2) above with respect to the lower bound of integration $a$ becomes...

$$\frac{\delta}{\delta a}\,g(z,m,v,a,b)=-\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\bigg(-(-a)-m\bigg)^2\right\}=-\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\bigg(a-m\bigg)^2\right\} \tag{27}$$

**D.** Using Equation (27) above the equation for the second derivative of Equation (2) above with respect to the lower bound of integration $a$ is...

$$\frac{\delta^2}{\delta a^2}\,g(z,m,v,a,b)=\frac{\delta}{\delta a}\Bigg(-\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\bigg(a-m\bigg)^2\right\}\Bigg)=-\frac{\delta}{\delta a}\Bigg(\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\bigg(a-m\bigg)^2\right\}\Bigg) \tag{28}$$

Using Equation (20) above as our guide the solution to Equation (28) above is...

$$\frac{\delta^2}{\delta a^2}\,g(z,m,v,a,b)=\frac{a-m}{v}\sqrt{\frac{1}{2\pi v}}\,\text{Exp}\left\{-\frac{1}{2v}\bigg(a-m\bigg)^2\right\} \tag{29}$$